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Bounds for the Solution of the Linear Self-Adjoint Second Order Differential Equation

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Walter Leighton [1, 2] studied the boundedness of the ordinary self-adjoint second order differential equation,

$$(py')' + qy = 0 \tag{1}$$

on the semi-infinite positive interval $a \leqslant x$ through the use of a quadratic form,

$$V = (py')^2 + pqy^2. \tag{2}$$

In his work Leighton assumed that p, q and $(pq)'$ were continuous with p and q positive for $x \geqslant a$. His arguments depend on the sign of $(pq)'$.

In this note it is shown how this quadratic form and another closely related one,

$$U = y^2 + \frac{1}{pq}(py')^2, \tag{3}$$

can be used to determine bounds on y and py' on the finite interval $[a, b]$ in in which $(pq)'$ is not necessarily of one sign. The bounds are expressible in terms of p and q at a or b and their values at the points where (pq) change from monotone increasing to monotone decreasing or vice versa. It is assumed below that p and q are positive and absolutely continuous. This assumption guarantees the existence of p' and q' almost everywhere in $[a, b]$. It is also assumed that the interval $[a, b]$ can be partitioned into N finite subintervals such that in each interval the product pq is either monotone increasing or monotone decreasing.

Let I be such a subinterval, then it is straightforward to show that the quadratic form

$$V = (pq)\,y^2 + (py')^2$$

is monotone increasing (decreasing) on I if and only if (pq) is monotone increasing (decreasing) on I, and furthermore the quadratic form

$$U = y^2 + \frac{1}{(pq)} (py')^2 \tag{3}$$

is monotone increasing (decreasing) on I if and only if pq is monotone decreasing (increasing) on I. In each subinterval then there is both a monotone increasing and a monotone decreasing quadratic form. Also then in each subinterval bounds on either y^2 or $(py')^2$ can be found in terms of values of U and V at one of the end points of the interval.

Let I_1 and I_2 be two adjacent subintervals with I_2 to the right of I_1, with x_1 the common point. Let W_1 be the monotone decreasing quadratic form in I_1 and W_2 be the monotone decreasing quadratic form in I_2. Then if a positive constant C can be found such that

$$W_2(x_{1+}) \leqslant CW_1(x_{1-}) \tag{4}$$

it is seen that then bounds can be determined on y^2 and $(py')^2$ in both I_1 and I_2 in terms of value of y^2 and $(py')^2$ at the left end point of I_1.

Continuing in this manner, it is possible to determine bounds on y^2 and $(py')^2$ in terms of $y^2(a)$ and $(p(a)\, y'(a))^2$ throughout the interval I.

Similar statements hold cencerning the right end point of the interval I, b except that now the appropriate monotone increasing function is used, and the appropriate inequality is

$$W_1(x_{1-}) \leqslant CW_2(x_{1+}). \tag{5}$$

These constants will be determined next. Use is made here of the fact that since p and q are of bounded variation it follows that y and py' are continuous on I. Furthermore, the constant C is determined so that if pq is continuous at x_1 then the equalities in Eqs. (4) and (5) hold.

It appears that these inequalities must be developed case by case. Two constants will enter the analysis

$$C_1 = \max\,(pq_+\,, pq_-) \tag{6}$$

$$C_2 = \max\left(\frac{1}{pq_+}\,, \frac{1}{pq_-}\right), \tag{7}$$

where the $+$ denotes the limit from the right and the $-$ the limit from the left. When making deductions from left to right the constant C must be determined which satisfies Eq. (4) with the equality holding only if $pq_+ = pq_-$. If pq is m.i. in I_1 and m.d. in I_2 then

$$C = C_1\,. \tag{8}$$

If pq is m.d. in I_1 and m.i. in I_2 then

$$C = C_2 . \tag{9}$$

When making deductions from right to left, the constant C must be determined which satisfies Eq. (5) with the equality holding only if $pq_+ = pq_-$. If pq *is* m.i. in I_1 and m.d. in I_2 then

$$C = C_1 . \tag{10}$$

If pq is m.d. in I_1 and m.i. in I_2 then

$$C = C_2 . \tag{11}$$

To illustrate the result two examples are given and they are compared with a result given in a lemma by Bellman [3].

The first example is the simple case

$$p(x) = q(x) = 1, \qquad 0 \leqslant x \leqslant 1 \tag{12}$$

with the boundary conditions,

$$y(0) = 1, \qquad y'(0) = 0.$$

Using the method presented in this paper it is seen that

$$| y(x) | \leqslant | y(0) | = 1 \qquad 0 \leqslant x \leqslant 1. \tag{13}$$

The bound deducible from Bellman's lemma is found from the equivalent Volterra integral equation formulation of the differential equation in (1), namely,

$$y(x) = y(0) + p(0) y'(0) r(x) - \int_{x'=0}^{x} q(x') [r(x) - r(x')] y(x'), \tag{14}$$

where

$$r(x) = \int_{x''=0}^{x} 1/p(x'').$$

Using Bellman's lemma the following bound can be deduced

$$y(x) \leqslant m_1 \exp \left\{ \frac{1}{m_2} \int_{x'=0}^{x} q(x') \right\}, \qquad 0 \leqslant x \leqslant 1 \tag{15}$$

where

$$m_1 \geqslant y(0) + p(0) y'(0) r(x), \qquad 0 \leqslant x \leqslant 1$$
$$m_2 \leqslant p(x), \qquad 0 \leqslant x \leqslant 1.$$

For this first example, Eq. (15) yields the estimate

$$y(x) < \exp x. \tag{16}$$

This result is similar to that obtainable from Brauer's [4] work and is not as tight as that obtained in Eq. (13).

However, it should be pointed out that the assumptions necessary to obtain the Bellman's and Brauer's result as well as the generalization of Bellman's lemma contained in Coddington and Levinson [5] are not as stringent as those in this paper.

Bellman (see [3], p. 138) also gives another result which should be mentioned. For the case $p(x) = 1$, $| q'(x) |$ integrable and $q(x) > 0$

$$y^2(x) < \frac{m}{q(x)} \exp \int_{x'=0}^{x} \frac{| q'(x) |}{q(x)}, \tag{17}$$

however, the determination of the constant m requires knowledge of $| y'(x) |$.

For the first example Eq. (17) yields a bound similar to Eq. (13), although without additional information the size of m is unknown.

One aspect common to the above methods of Bellman and Brauer is that they all involve terms of the form

$$\exp g(x)$$

where $g(x)$ is positive and monotone increasing. The bounds developed in this paper do not have this disadvantage.

For a second example, $p(x)$ and $q(x)$ are defined as follows

$$\begin{aligned} p(x) &= 1 + 4x, && 0 \leqslant x < \tfrac{1}{4} \\ &= 2, && \tfrac{1}{4} \leqslant x \leqslant \tfrac{3}{4} \\ &= 1 + 4(1 - x), && \tfrac{3}{4} \leqslant x \leqslant 1, \end{aligned} \tag{18}$$

and

$$\begin{aligned} q(x) &= 2, && 0 \leqslant x < \tfrac{1}{2} \\ &= 1 + 6(1 - x), && \tfrac{1}{2} < x \leqslant 1. \end{aligned} \tag{19}$$

Note that $q(x)$ is discontinuous at $x = \frac{1}{2}$. The produce pq is m.i. for $0 \leqslant x < \frac{1}{2}$ and m.d. for $\frac{1}{2} < x < 1$. The boundary conditions will be taken the same as in the first example.

Bellman's lemma yields the following estimate

$$y(x) \leqslant \exp \int_{x'=0}^{x} q(x') \tag{20}$$

since for this example m_2 is unity. The results of this paper show that

$$
\begin{aligned}
y(x) &\leqslant 1, && 0 \leqslant x \leqslant \tfrac{1}{2} \\
&\leqslant \sqrt{\frac{8}{p(x)\,q(x)}}, && \tfrac{1}{2} \leqslant x \leqslant 1.
\end{aligned} \tag{21}
$$

The smallest value of the product pq for $\frac{1}{2} < x \leqslant 1$ is 2. Here again the result is tighter than that obtainable with Bellman's lemma.

These techniques can also be applied to determine bounds on the moduli of the electric and magnetic fields in a one-dimensionally stratified layer for a monochromatic plane, wave incident. Let the modulus of the electric field be denoted by p_1 and that of the magnetic field by p_2. Then it can be shown from Maxwell's equations that

$$V = \frac{\epsilon(x)}{\mu(x)} p_1^2 + p_2^2$$

is m.i. (m.d.) if and only if the quotient ϵ/μ is m.i. (m.d.) where $\epsilon(x)$ denotes the permittivity and $\mu(x)$ denotes the permeability.

Similarly, the quadratic form

$$V = p_1^2 + \frac{\mu(x)}{\epsilon(x)} p_2^2$$

is m.i. (m.d.) if and only if the quotient $\epsilon(x)/\mu(x)$ is m.d. (m.i.). It is assumed that $\epsilon(x)$ and $\mu(x)$ have the same properties previously assumed for $p(x)$ and $q(x)$.

Estimates of the magnitude of p_1 and p_2 at the two end points of the layer are obtained by conservation of energy principles.

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